Fakultät für Mathematik
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<u>Examination in Mathematics I</u>

(15.02.2002)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- printed script "Mathematics for Students of Economics and Management"

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	5	8	9	10	11	7	50

Problems:

1. Prove by induction

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}.$$

2. (a) A firm produces 10,000 video recorders in its first year 1981.

    i. How many video recorders does the firm produce up to the end of n years if production increases by 20% every year?
    Specify the result for $n = 20$.

    ii. Which was the first year with a production of more than 100,000 video recorders?

   (b) Check whether the series

$$\sum_{k=1}^{\infty} \frac{2k^2}{(k+1) \cdot 3^k}$$

   converges. Use the quotient criterion.

3. Let

$$P(x) = x^4 - 2x^2 - 8.$$

   (a) Determine all real and complex zeros of $P(x)$.

   (b) Find the polar (trigonometric) forms of the complex zeros.

   (c) Is $P(x)$ an even or odd function?

   (d) Is $P(x)$ a bijective function? Does the inverse function P^{-1} exist? Give reason(s) for your statement.

4. (a) A function is given by the formula

$$f_1(x) = \ln(x^2 - x + 1) + 2; \quad x \in \mathbb{R}$$

   Find intervals of monotonicity and concavity/convexity.

   (b) Another function $f_2(x)$ is given by

$$f_2(x) = \begin{cases} \ln(x^2 - x + 1) + 2 & \text{for } \ x < 0 \\ ax + b & \text{for } \ x \geq 0 \end{cases}$$

   Find a and b so that $f_2(x)$ is continuous and differentiable at all $x \in \mathbb{R}$.

5. Let

$$f_t(x) = 1 - \frac{2e^x}{e^x + t}; \quad t > 0$$

be a class of functions depending on x with a constant parameter t.

(a) Find domain, zeros and $\lim_{x \to \pm\infty} f_t(x)$.

(b) Determine extreme points and inflection points of $f_t(x)$.

(c) Let $t = e$. Decide whether the function $f_e(x)$ is elastic at $x = 2$.

6. (a) Find

$$\int \frac{x^3 \, dx}{\sqrt{1 - x^2}}.$$

(b) Evaluate

$$F(T) = \int_0^T a e^{-at} \, dt$$

and show that $\lim_{T \to \infty} F(T) = 1$.

Examination in Mathematics I - Solutions

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1. $$\begin{array}{lll} A(1) & : & \frac{1}{2} = 2 - \frac{3}{2} \\ A(n) & : & \sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n} \\ A(n+1) & : & \sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+3}{2^{n+1}} \\ A(n) \Rightarrow A(n+1) & : & \sum_{k=1}^{n+1} \frac{k}{2^k} = \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}} \\ & & = 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \\ & & = 2 - \frac{2n+4-n-1}{2^{n+1}} \\ & & = 2 - \frac{n+3}{2^{n+1}} \end{array}$$

2. (a) i. $s_n = a_1 \dfrac{1-q^n}{1-q}$ with $a_1 = 10{,}000$ and $q = 1.2$; $\quad s_{20} = 1{,}866{,}879.9$

   ii. 1994 was the first year.

   (b) $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{1}{3} < 1$; series converges.

3. (a) $x_1 = 2, \quad x_2 = -2, \quad x_3 = \sqrt{2}\, i, \quad x_4 = -\sqrt{2}\, i$

   (b) $x_3 = \sqrt{2}\,(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}), \quad x_4 = \sqrt{2}\,(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$

   (c) $P(x)$ is even.

   (d) $P(x)$ is not bijective and the inverse function does not exist. An even function is not injective.

4. (a) $$\begin{array}{lll} -\infty < x \le \frac{1}{2} & : & f'(x) \le 0, \quad f(x) \text{ decreasing} \\ \frac{1}{2} \le x < \infty & : & f'(x) \ge 0, \quad f(x) \text{ increasing} \\ & & \\ -\infty < x \le \frac{1}{2} - \frac{\sqrt{3}}{2} & : & f''(x) \le 0, \quad f(x) \text{ concave} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \le x \le \frac{1}{2} + \frac{\sqrt{3}}{2} & : & f''(x) \ge 0, \quad f(x) \text{ convex} \\ \frac{1}{2} + \frac{\sqrt{3}}{2} \le x < \infty & : & f''(x) \le 0, \quad f(x) \text{ concave} \end{array}$$

   (b) $a = -1; \quad b = 2$

5. (a) $D_f = \mathbb{R}$; zero: $x_0 = \ln t$; $\lim\limits_{x\to\infty} f_t(x) = -1$; $\lim\limits_{x\to-\infty} f_t(x) = 1$
   (b) no extreme points; inflection point at $(\ln t, 0)$
   (c) $f_e(x)$ is elastic at $x = 2$

6. (a)
$$\int \frac{x^3\,dx}{\sqrt{1-x^2}} = -(1-x^2)^{\frac{1}{2}} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$
   (b)
$$F(T) = \int\limits_0^T ae^{-at}\,dt = -e^{-aT} + 1$$
$$\lim_{T\to\infty} F(T) = 1 \quad \text{for} \quad a > 0$$